SOME DEFINITIONS

Vector space
The space of all vectors, multiplied by a scalar or added

Basis
Any set of n linearly independent vectors is called the basis for the vector space

Vector geometry
Vectors can be represented geometrically as a point in a *p*-dimensional space and as a vector originating from the origin of that space. Every vector can be defined in terms of its length and direction.

Vector length
The length of a vector is a result of the Pythagorean theorem: Length of $\underline{x}$ = $L(\underline{x}) = \sqrt{\underline{x}'\underline{x}}$.

Vector angle
Two vectors can be described in terms of the angle between them, θ, such that $\text{Cos}(\theta) = \dfrac{\underline{x}'_1 \underline{x}_2}{L(\underline{x}_1)L(\underline{x}_2)}$.

Linear combination
Vector $\underline{y} = a_1\underline{x}_1 + a_2\underline{x}_2 + \ldots + a_p\underline{x}_p$ is a linear combination of vectors $\underline{x}_p$. The set of all linear combinations of vectors $\underline{x}_p$ is called the linear span.

Linear dependence
A set of vectors $\underline{x}_p$ is linearly dependent if there exists a set of scalars, not all zero, such that:
$a_1\underline{x}_1 + a_2\underline{x}_2 + \ldots + a_p\underline{x}_p = \underline{0}$

Determinant
A summary scalar of a square matrix.

Properties of the determinant

1. $|\mathbf{A}| = |\mathbf{A}'|$
2. If each element of a row or column of $\mathbf{A}$ is zero, then $|\mathbf{A}| = 0$
3. If any two rows or columns are identical, then $|\mathbf{A}| = 0$
4. If $\mathbf{A}$ is nonsingular, then $|\mathbf{A}| = |\mathbf{A}|\ |\mathbf{A}^{-1}| = 1$
5. $|\mathbf{AB}| = |\mathbf{A}|\ |\mathbf{B}|$

Row rank
The maximum number of linearly independent rows, considered row vectors

Column rank
The maximum number of linearly independent columns, considered column vectors

Rank of a matrix
The row rank or the column rank, since they are equal

Inverse
The inverse of a matrix (e.g., $\mathbf{A}$), is another matrix ($\mathbf{A}^{-1}$) such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. The inverse is only defined for square matrices. If a square matrix has an inverse, the inverse is unique.

An inverse of a square matrix exists if the rank of the matrix is equal to the order of the matrix. Such a matrix is said to be *nonsingular* or have *full rank*. A square matrix with rank less than its order is said to be singular or deficient rank and does not have an inverse.

Nonsingular
A square matrix with rank equal to the number of rows or columns. This implies that the columns of the matrix are linearly independent. The inverse exists for a nonsingular matrix, such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

For a square matrix, the following statements are equivalent

1. $\mathbf{A}$ is nonsingular
2. $|\mathbf{A}| \neq 0$
3. The inverse exists

Trace
A summary scalar containing information about the diagonal of a square matrix, the sum of the diagonal elements.

Orthogonal Vectors
Vectors are considered orthogonal if they intersect perpendicularly; vectors are separated by 90^0.

Orthogonal Matrix
A square matrix is orthogonal if its rows are separated by 90^0 and have unit lengths, such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. An orthogonal matrix also has the property $\mathbf{A}\mathbf{A}' = \mathbf{A}'\mathbf{A} = \mathbf{I}$.
So the columns are also separated by 90^0 and of unit lengths.

Eigenvalues
Also called the characteristic roots of a matrix, these are scalar values that satisfy a polynomial equation $| \mathbf{A} - \lambda \mathbf{I} | = 0$.

Univariate Normal Density

The univariate normal distribution has a mean μ and variance σ^2, denoted by $N(\mu, \sigma^2)$, with the probability density function:

$$f(\underline{x}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}[(x-\mu)/\sigma]^2}$$

Distance

The exponent of the univariate normal density function measures the square of the distance from x to μ in standard deviation units. This can be generalized to a p x 1 vector $\underline{x}$ of observations on several variables for the multivariate case as:

$$(\underline{x} - \underline{\mu})'\boldsymbol{\Sigma}^{-1}(\underline{x} - \underline{\mu})$$

This is the square of the generalized distance from $\underline{x}$ to $\underline{\mu}$. This is used to generalize the normal density function to the multivariate case.

Multivariate Normal Density

A *p*-dimensional normal density is noted by $N_p(\underline{\mu}, \boldsymbol{\Sigma})$ with the form:

$$f(\underline{x}) = \frac{1}{(2\pi)^{p/2}\left|\Sigma\right|^{1/2}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu})}$$

Properties of vector $\underline{x} \sim N_p(\underline{\mu}, \boldsymbol{\Sigma})$

1. linear combinations of the components of $\underline{x}$ are normally distributed
2. all subsets of the components of $\underline{x}$ have multivariate normal distribution
3. zero covariance suggests that corresponding components are independently distributed
4. conditional distributions of the components are multivariate normal

Assessing Assumption of Normality

Most multivariate techniques assume each vector observation $\underline{x}$ (by person) comes from a multivariate normal distribution. The quality of the inferences made by multivariate methods depends on the degree to which the population conforms to the multivariate normal distribution.

When the parent population is multivariate normal and both n and $n - p$ are greater than 30, each of the squared distances $d_1^2, d_2^2, ..., d_n^2$ should behave like a chi-square random variable.

A Q-Q plot (based on chi-square distribution with p degrees of freedom) should result in a straight line through the origin with a slope = 1. A systematically curved pattern would indicate departure from normality. Points far above the line (furthest from the origin) would indicate large distances or outliers.

To assess the distances for each observation j, the squared distances can be computed as

$$\underline{d}_j^2 = (\underline{x}_j - \underline{\bar{x}})'S^{-1}(\underline{x}_j - \underline{\bar{x}}).$$

This is known as the Mahalnobis Distance

In SPSS Matrix commands, this can be obtained by computing distance as:

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Compute distance = diag(d*inv(sigma)*T(d)).
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Once these values are saved, they can be plotted in a Q-Q plot based on a Chi-Square distribution with p degrees of freedom.